M.Sc. - Mathematics
I Semester End Examination - May 2022
Real Analysis

Course Code: MM102T **QP Code: 11002**
Time: 3 hours **Total Marks: 70**

Instructions: 1. Answer any FIVE questions
2. All questions carry **equal** marks

1 a) Evaluate $\int_0^3 x d\{[x]\}$, where $[x]$ is the maximum integer function.
b) If $f(x) \in R[\alpha(x)]$ on $[a,b]$ then prove that $-f(x) \in R[\alpha(x)]$ on $[a,b]$.
c) If $f(x) \in R[\alpha(x)]$ on $[a,b]$ and $|f| \leq M$, then prove that $\left|\int_a^b f d\alpha\right| \leq M[\alpha(b) - \alpha(a)]$.

(4+5+5)

2 a) Assuming $f(x)$ is monotonic on $[a,b]$ and $\alpha(x)$ is monotonically increasing and continuous function on $[a,b]$ prove that $f \in R(\alpha)$ on $[a,b]$.

b) If $f \in R(\alpha)$ on $[a,b]$, $f \in [m,M]$ and φ is continuous function of f on $[m,M]$ then prove that $\varphi(f(x)) \in R(\alpha)$ on $[a,b]$.

c) Give an example of a function f such that $|f| \in R(\alpha)$ on $[0,1]$ and $|f| \notin R(\alpha)$ on $[0,1]$.

(5+7+2)

3 a) Consider the function $\beta_1(x)$ and $\beta_2(x)$ defined as follows:

$$\beta_1(x) = \begin{cases} 0, when\ x \leq 0 \\ 1, when\ x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, when\ x < 0 \\ 1, when\ x \geq 0 \end{cases}$$

Verify whether $\beta_1(x) \in R[\beta_2(x)]$ on $[-1,1]$.

b) If f and φ are continuous on $[a,b]$ and φ is strictly increasing on $[a,b]$ and Ψ is an inverse function of φ, then prove that $\int_a^b f(x)dx = \int_{\phi(a)}^{\phi(b)} f(\psi(\varphi))d\psi(\varphi)$.

c) Show that a function of bounded variation on $[a,b]$ is bounded. **(7+4+3)**

4 a) State and prove Weierstrass M- test.
b) Show that $\{e^{-nx}\}$ is uniformly convergent on $[a,b]$.

c) Suppose $f_n \to f$ uniformly on $[a,b]$ and if $x_0 \in [a,b]$ such that

$$\lim_{x \to x_0} f_n(x) = A_n, \quad n = 1,2,3, \ldots \text{ then prove that}$$

i) A_n convergent ii) $\lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} f_n(x)$. **(5+4+5)**

5 a) Prove the necessary and sufficient condition for a sequence $\{f_n\}$ of functions defined on a set S to be uniformly convergent is that for each $\in > 0$ there corresponds on such that $\forall n \geq m$, $\forall p \geq 0$ and $\forall x \in S$, $|f_{n+p}(x) - f_n(x)| < \in$.

b) Let $\{f_n(x)\}$ be a sequence of functions uniformly converges to $f(n)$ on $[a, b]$ and each $f_n(x) \in R[a, b]$. Then prove that $f(n) \in R[a, b]$, also prove that $\lim_{n\to\infty} \int_a^x f_n(t)dt = \int_a^x f(t)dt \ \forall x \in [a, b]$.

c) Show that $\sum_{n=1}^{\infty} nxe^{-nx^2}$ converges pointwise and not uniformly on $[0, k]$, $k > 0$.

(5+5+4)

6 a) State and prove Heine-Borel theorem.

b) Define a K-cell in R^k. Let $I_1 \supset I_2 \supset I_3 \supset \cdots$ be a sequence of K-cell in R^k show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. **(8+6)**

7 a) Let $E \subset R^n$ be an open set $f: E \to R^m$ is a map. Prove that if f is continuously differentiable if and only if the partial derivatives $D_j f_i$ exists and continuous on E for $1 \leq i \leq m, 1 \leq j \leq n$.

b) If $T_1, T_2 \in L(R^n, R^m)$, then prove that

i) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

ii) $\|\alpha T_1\| = |\alpha| \|T_1\|$

c) Discuss the continuity of the function on R^2 of

$$f(x, y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$$

(7+4+3)

8. State and prove the implicit function theorem. **(14)**

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